

# THE WIGNER CAUSTIC ON SHELL AND SINGULARITIES OF ODD FUNCTIONS

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**ABSTRACT.** We study the Wigner caustic on shell of a Lagrangian submanifold  $L$  of affine symplectic space. We present the physical motivation for studying singularities of the Wigner caustic on shell and present its mathematical definition in terms of a generating family. Because such a generating family is an odd deformation of an odd function, we study simple singularities in the category of odd functions and their odd versal deformations, applying these results to classify the singularities of the Wigner caustic on shell, interpreting these singularities in terms of the local geometry of  $L$ .

## 1. INTRODUCTION

The Wigner caustic of a smooth convex closed curve  $L$  on affine symplectic plane was first introduced by Berry, in his celebrated 1977 paper [3] on the semiclassical limit of Wigner's phase-space representation of quantum states. Thus, when  $L$  is the classical correspondence of a pure quantum state, the Wigner function of this state takes on high values, in the semiclassical limit, at points in a neighborhood of  $L$  and also in a neighborhood of a singular closed curve in its interior, generically formed by an odd number of cusps: the Wigner caustic of  $L$ .

Some years later, Ozorio de Almeida and Hannay studied the Wigner caustic of a smooth Lagrangian torus  $L$  on affine symplectic 4-space [13]. Since their main object of study was the geometrical place where the amplitude of the Wigner function of the pure quantum state corresponding to  $L$  rises considerably, in the semiclassical limit, they considered  $L$  itself as part of the Wigner caustic and focused some attention on the part of the Wigner caustic that is close to and contains  $L$ .

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From a purely geometrical point of view, the Wigner caustic of  $L$ , hereby denoted  $\mathbf{E}_{1/2}(L)$ , is defined as the locus of midpoints of segments connecting pairs of points on  $L$  with “parallel” affine tangent spaces. Here, parallelism is taken in a broad sense, also allowing for *weak* parallelism, when the direct sum of the tangent spaces of  $L$  at the two points do not span the whole  $\mathbb{R}^{2m}$ . However, as mentioned above, from the perspective of applications of Wigner caustics in quantum physics, it is interesting to consider an even broader definition of parallelism, when a single point of  $L$  is identified as a pair of points with parallel affine tangent spaces (in this case *strongly* parallel spaces). Then, with this extended notion in the geometrical definition, the submanifold  $L$  itself is a subset of  $\mathbf{E}_{1/2}(L)$ . The part of  $\mathbf{E}_{1/2}(L)$  that is close to  $L$  and that contains  $L$  is called the *Wigner caustic on shell*.

In this paper, we study the Wigner caustic on shell of a smooth Lagrangian submanifold  $L$  of the affine symplectic space  $(\mathbb{R}^{2m}, \omega)$ , focusing on its Lagrangian-stable singularities when  $L$  is a curve or a surface. Its definition in terms of a generating family reveals the fact that the Wigner caustic on shell has a (hidden) symmetry under the action of  $\mathbb{Z}_2$ , because its generating family is an odd deformation of an odd function of the variables. No such symmetry exists for the part of the Wigner caustic that is away from  $L$ , whose simple stable Lagrangian singularities have been studied in a previous paper [7].

Now, our interest in studying singularities of the Wigner caustic stems from semiclassical dynamics. Because the amplitude of the Wigner function rises sharply along the Wigner caustic, in the semiclassical limit, there is where uniform asymptotic expressions must be used. However, the kind of uniform asymptotic expression for the semiclassical Wigner function in a neighborhood of a point varies according to the kind of singularity of the Wigner caustic at that point [3]. Thus, for a finer treatment of the dynamics of the semiclassical Wigner function of a pure quantum state [15], it is important to classify the singularities of the Wigner caustic (off and on shell) of a Lagrangian submanifold, which are stable under the group of symplectomorphisms of  $(\mathbb{R}^{2m}, \omega)$ .

Because such singularities are described by generating families, here we focus attention on simple singularities of function-germs (simple here in the classical notion of absence of modal parameters [1]) and their versal deformations. Thus, for the Wigner caustic on shell, our first aim is to obtain the list of all simple singularities in the category of odd-functions. This paper is, therefore, divided in three parts.

The first part, Section 2, presents the motivation and definition of the Wigner caustic on shell of a Lagrangian submanifold.

The second part, Section 3, is independent of the other sections and is devoted to the classification of simple singularities of odd functions and their odd deformations. By odd function-germs at  $0 \in \mathbb{R}^m$  we mean  $\mathbb{Z}_2$ -equivariant smooth function-germs, with  $\mathbb{Z}_2$  action on the source:  $(x_1, \dots, x_m) \mapsto (-x_1, \dots, -x_m)$  and on the target:  $y \mapsto -y$ . We classify odd function-germs using classical  $\mathcal{R}$ -equivalence (composition with germs of diffeomorphisms on the source) restricted to the subgroup of odd diffeomorphism-germs, which is natural in this context. We prove there are no simple odd singularities if the dimension of the source is greater than two and classify all simple odd function-germs in dimensions one and two, presenting their odd mini-versal deformations. Although this could be considered as a classical subject in singularity theory, surprisingly no such classification list of simple odd singularities has been found by the authors in the literature.

In one variable the simple odd singularities are of type that we shall denote  $A_{2k/2}$ , which have codimension  $k$  in the category of odd function-germs and which coincide with an intersection of the classical  $\mathcal{R}$ -orbit of  $A_{2k}$  singularities of codimension  $2k$  with the module of odd function-germs. In two variables, the simple odd singularities are divided in two groups: the first one of types hereby denoted  $D_{2k/2}^\pm$  and  $E_{8/2}$ , of odd codimensions  $k$  and 4 respectively, which are the intersections of classical  $\mathcal{R}$ -orbits of types  $D_{2k}^\pm$  and  $E_8$ , of codimensions  $2k$  and 8 respectively, with the module of odd function-germs. The second group consists of the singularities of types hereby denoted  $J_{10/2}^\pm$  and  $E_{12/2}$ , of respective odd codimensions 5 and 6, these notations chosen because they are  $\mathcal{R}$ -equivalent to singularities  $J_{10}$  and  $E_{12}$  of respective codimensions 10 and 12, these later being unimodal in Arnold's classification.

The third part, Section 4, applies the results of Section 3. For Lagrangian curves, we give the conditions for realizing the odd deformations of singularities  $A_{2/2}$  and  $A_{4/2}$  as generating families for simple stable Lagrangian singularities of the Wigner caustic on shell, and describe these singularities. For Lagrangian surfaces, we present the realization conditions for the singularities of the Wigner caustic on shell of types  $D_{2k/2}^\pm$ ,  $k = 2, 3, 4$ , and  $E_{8/2}$ . Because the odd codimension in this context can be at most 4, these are all the simple singularities that can be realized as simple stable Lagrangian singularities of the Wigner caustic on shell. Finally, we also interpret the realization condition of each of these singularities of the Wigner caustic on shell in terms of the local geometry of the Lagrangian curve or the Lagrangian surface.

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## 2. THE WIGNER CAUSTIC ON SHELL

**2.1. Physical origins of the Wigner caustic on shell.** The following presentation is sketchy and can be found expanded in various textbooks and research papers (see [3, 13, 15], for instance).

We recall that, in non-relativistic quantum mechanics, a *pure state of the system* is usually defined as a normalized vector  $\Psi$  in a Hilbert space  $\mathcal{H}$ . In many simple cases,  $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^m)$ , the space of complex-valued square-integrable functions on  $\mathbb{R}^m$ . Here,  $\mathbb{R}^m$  is commonly interpreted either as the *configuration-space*  $Q$  or the *momentum-space*  $P$  and  $m \in \mathbb{N}$  is the number of *degrees of freedom* of the system.

The Fourier transform  $\mathcal{F} : L^2_{\mathbb{C}}(\mathbb{R}^m) \rightarrow L^2_{\mathbb{C}}(\mathbb{R}^m)$  relates configuration-space and momentum-space representations of a state  $\Psi$ , by

$$\psi(q) \mapsto \mathcal{F}_\psi(p) = \frac{1}{(2\pi\hbar)^m} \int_{\mathbb{R}^m} \psi(q) \exp(ipq/\hbar) dq ,$$

where  $i = \sqrt{-1}$  and  $\hbar$  is a positive constant, called Planck's constant, which provides a scale for comparing quantum to classical phenomena.

On the other hand, in classical conservative dynamics, the concept of a *phase-space*  $\Pi$  is predominant. In the simple cases when  $Q = P = \mathbb{R}^m$ ,  $\Pi = P \times Q = \mathbb{R}^{2m}$ , endowed with the symplectic form  $\omega = \sum_{i=1}^m dp_i \wedge dq_i$ , is an affine-symplectic space.

The Wigner transform  $\mathcal{W} : L^2_{\mathbb{C}}(\mathbb{R}^m) \rightarrow L^1_{\mathbb{R}}(\mathbb{R}^{2m}, \omega)$  defines a phase-space representation of a pure state  $\Psi$ , called its *Wigner function*, from the configuration-space representation of  $\Psi$ , by

$$\psi(q) \mapsto \mathcal{W}_\psi(p, q) = \frac{1}{(\pi\hbar)^m} \int_{\mathbb{R}^m} \psi^*(q - \zeta) \psi(q + \zeta) \exp(2ip\zeta/\hbar) d\zeta .$$

The Wigner function satisfies reality and Liouville-normalization,

$$\mathcal{W}_\psi(p, q) = \mathcal{W}_\psi^*(p, q) , \quad \int_{\mathbb{R}^{2m}} \mathcal{W}_\psi(p, q) dp dq = 1 , \quad dp dq = \omega^m / m!$$

and, although  $\mathcal{W}_\psi(p, q)$  can be negative, its partial integrals are not,

$$\int_{\mathbb{R}^m} \mathcal{W}_\psi(p, q) dp = |\psi(q)|^2 \geq 0 , \quad \int_{\mathbb{R}^m} \mathcal{W}_\psi(p, q) dq = |\mathcal{F}_\psi(p)|^2 \geq 0 ,$$

so that  $\mathcal{W}_\psi$  can be seen as a pseudo probability distribution on phase-space  $(\mathbb{R}^{2m}, \omega)$ , while  $|\psi|^2$  and  $|\mathcal{F}_\psi|^2$  are actual probability distributions on configuration-space and momentum-space, respectively.

In various instances, one is mostly interested in a pure state  $\Psi$  which is eigenstate of one or more self-adjoint operators on  $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^m)$ . If  $F \in \mathcal{B}(\mathcal{H})$  is self-adjoint, its classical correspondence is a real function  $f \in C^\infty_{\mathbb{R}}(\mathbb{R}^{2m}, \omega)$  so that, if  $F(\Psi) = \alpha\Psi, \alpha \in \mathbb{R}$ , then  $\Psi$  corresponds

classically to the level set  $\Lambda = \{x = (p, q) \in \mathbb{R}^{2m} : f(x) = \alpha\}$ , which for many values of  $\alpha$  is a smooth hypersurface in phase-space (a smooth Lagrangian curve  $\Lambda = L$  for systems with one degree of freedom).

For systems with  $m > 1$  degrees of freedom, two linearly independent functions  $f_1, f_2 \in \mathcal{C}_{\mathbb{R}}^{\infty}(\mathbb{R}^{2m}, \omega)$  are said to be in *involution* if  $X_{f_1}(f_2) = X_{f_2}(f_1) = 0$ , where  $X_{f_j}$  is the vector field defined by Hamilton's equation  $df_j + X_{f_j} \lrcorner \omega = 0$ . If there exist  $m$  linearly independent functions  $f_j$  in mutual involution, the classic dynamical system is integrable and each level set  $L = \{x \in \mathbb{R}^{2m} : f_j(x) = \alpha_j \in \mathbb{R}, j = 1, \dots, m\}$  is a Lagrangian submanifold of  $(\mathbb{R}^{2m}, \omega)$ . Such  $L$  may correspond to a pure state  $\Psi$  which is eigenstate of  $m$  linearly independent commuting self-adjoint operators  $F_j \in \mathcal{B}(\mathcal{H})$ ,  $[F_i, F_j] = 0$ ,  $F_j(\Psi) = \alpha_j \Psi$ .

The semiclassical approximation of  $\Psi$  can be formally seen as the asymptotic expansion on  $\hbar \ll 1$  of some representation of  $\Psi$ . Let's start with the crude expression for the semiclassical approximation of the Wigner function of a pure state in one degree of freedom [3]:

$$(2.1) \quad \mathcal{W}_{\psi}(x) \approx \sum_k \mathcal{A}_k^{\hbar}(x) \cos(S_k(x)/\hbar - \pi/4) ,$$

where  $S_k(x)$  is the symplectic area enclosed by the curve  $L = \{x' \in \mathbb{R}^2 : f(x') = \alpha\}$  and the  $k$ -th chord connecting two points  $x_k^+$  and  $x_k^-$  on  $L$ , whose midpoint is  $x$  (for  $x$  close to  $L$ , such a chord is often unique, or does not exist). Each amplitude function  $\mathcal{A}_k^{\hbar}(x)$  in (2.1) satisfies

$$(2.2) \quad \mathcal{A}_k^{\hbar}(x) \propto \frac{1}{|\omega(X_f^{+k}(x), X_f^{-k}(x))|^{1/2}} ,$$

where  $X_f^{\pm k}(x)$  is the Hamiltonian vector field  $X_f$  evaluated at the endpoint  $x_k^{\pm} \in L$  of the  $k$ -th chord, parallel translated to its centre  $x$ .

The number of chords centered on  $x$  connecting pairs of points on  $L$  varies, as  $x$  varies, and its bifurcation set is given by

$$(2.3) \quad \mathbf{E}_{1/2}(L) = \{x \in \mathbb{R}^2 : \exists k \ \omega(X_f^{+k}(x), X_f^{-k}(x)) = 0\} .$$

It is clear from (2.3) that  $\mathbf{E}_{1/2}(L)$  can be defined as the set of midpoints of chords connecting points on  $L$  whose tangent vectors to  $L$  at these endpoints are parallel.  $\mathbf{E}_{1/2}(L)$  is called the *Wigner caustic* of  $L$  and is precisely the set where some  $\mathcal{A}_k^{\hbar}$  blows up to infinity, see (2.2).

In fact, in a neighborhood of  $\mathbf{E}_{1/2}(L)$ , the crude expression (2.1) is inappropriate and must be substituted by uniform approximations that do not blow up to infinity on  $\mathbf{E}_{1/2}(L)$  if  $\hbar \neq 0$  but, nonetheless, take on very high values at  $\mathbf{E}_{1/2}(L)$  for  $\hbar \ll 1$ . However, the kind of uniform approximation to be used will depend on the kind of singularity of the Wigner caustic. Thus, where the Wigner caustic corresponds to a

fold singularity, the uniform approximation of the Wigner function is written in terms of Airy functions but, where the Wigner caustic has cusp singularities, Pearcey functions must be used (see [3]).

Now, it is obvious from (2.3) that  $L \subset \mathbf{E}_{1/2}(L)$ , so that  $\mathcal{W}_\psi$  peaks at  $L$  for  $\hbar \ll 1$ . On the other hand, as  $x \rightarrow L$ ,  $S(x) \rightarrow 0$  and  $\nabla S(x) \rightarrow 0$ , so that  $\mathcal{W}_\psi$  is not highly oscillatory in a small neighborhood of  $L$ , for  $\hbar \ll 1$ . This contrasts sharply with the situation when  $x$  is far from  $L$  where, even if  $x \in \mathbf{E}_{1/2}(L)$ ,  $\mathcal{W}_\psi$  is highly oscillatory for  $\hbar \ll 1$  and tends on average to 0 in any small neighborhood of  $x$ , as  $\hbar \rightarrow 0$ . Thus, as  $\hbar \rightarrow 0$ , the pseudo probability distribution  $\mathcal{W}_\psi$  tends on average to the singular probability distribution which is zero everywhere but on  $L$ , where  $\mathcal{W}_\psi$  tends to infinity. In this way,  $L$  can be seen as the classical correspondence of the pure state  $\Psi$ .

The less oscillatory behavior of the Wigner function  $\mathcal{W}_\psi$  in a neighborhood of  $L$  makes it convenient to separate the Wigner caustic of  $L$  in a part which is away from  $L$  and another which is very close to  $L$  and contains  $L$ . This latter is called the *Wigner caustic on shell*.

The situation for integrable systems with more degrees of freedom is similar: the crude semiclassical expression for the Wigner function is

$$(2.4) \quad \mathcal{W}_\psi(x) \approx \sum_k \tilde{\mathcal{A}}_k^\hbar(x) \cos(\tilde{S}_k(x)/\hbar - n_k\pi/4),$$

where  $\tilde{S}_k(x)$  is the symplectic area of any surface bounded by a curve formed by taking any arc of the Lagrangian submanifold  $L = \{x' \in \mathbb{R}^{2m} : f_j(x') = \alpha_j, j = 1, \dots, m\}$  and closing it with the  $k$ -th chord connecting two points  $x_k^+$  and  $x_k^-$  on  $L$ , with midpoint  $x$ , and where

$$(2.5) \quad \tilde{\mathcal{A}}_k^\hbar(x) \propto \frac{1}{|\det[\omega(X_{f_i}^{+k}(x), X_{f_j}^{-k}(x))]|^{1/2}},$$

with  $X_{f_j}^{\pm k}(x)$  being the Hamiltonian vector field  $X_{f_j}$  evaluated at the endpoint  $x_k^\pm \in L$  of the  $k$ -th chord, parallel translated to its centre  $x$ . Also, the integer  $n_k$  in (2.4) is the signature of the  $m \times m$  matrix  $[\omega(X_{f_i}^{+k}(x), X_{f_j}^{-k}(x))]$ . Therefore, in this case,

$$(2.6) \quad \mathbf{E}_{1/2}(L) = \{x \in \mathbb{R}^{2m} : \exists k \quad \det[\omega(X_{f_i}^{+k}(x), X_{f_j}^{-k}(x))] = 0\}$$

and can be identified with the set of midpoints of chords connecting points on  $L$  whose tangent spaces to  $L$  at these endpoints are *weakly parallel*, in other words, do not span the whole  $\mathbb{R}^{2m}$ , see [13]. Again, uniform approximations must be used instead of (2.4) in a neighborhood of  $\mathbf{E}_{1/2}(L)$  and, for  $\hbar \ll 1$ ,  $\mathcal{W}_\psi$  is not highly oscillatory in a

small neighborhood of  $L$ , which is the classical correspondence of  $\Psi$ , and it is therefore natural to single out the Wigner caustic on shell.

## 2.2. Mathematical definition of the Wigner caustic on shell.

Let  $L$  be a smooth Lagrangian submanifold of the affine symplectic space  $(\mathbb{R}^{2m}, \omega = \sum_{i=1}^m dp_i \wedge dq_i)$ . Let  $a, b$  be points of  $L$  and let  $\tau_{a-b} : \mathbb{R}^{2m} \ni x \mapsto x + (a - b) \in \mathbb{R}^{2m}$  be the translation by the vector  $(a - b)$ .

**Definition 2.1.** A pair of points  $a, b \in L$  is a **weakly parallel** pair if

$$T_a L + \tau_{a-b}(T_b L) \neq \mathbb{R}^{2m}.$$

A weakly parallel pair  $a, b \in L$  is called  **$k$ -parallel** if

$$\dim(T_a L \cap \tau_{b-a}(T_b L)) = k.$$

If  $k = m$  the pair  $a, b \in L$  is called **strongly parallel**, or just **parallel**.

**Definition 2.2.** A **chord** passing through a pair  $a, b$ , is the line

$$l(a, b) = \{x \in \mathbb{R}^n : x = \eta a + (1 - \eta)b, \eta \in \mathbb{R}\}.$$

**Definition 2.3.** For a given  $\eta$ , an **affine  $\eta$ -equidistant** of  $L$ , denoted  $\mathbf{E}_\eta(L)$ , is the set of all  $x \in \mathbb{R}^{2m}$  s.t.  $x = \eta a + (1 - \eta)b$ , for all weakly parallel pairs  $a, b \in L$ . Note that, for any  $\eta$ ,  $\mathbf{E}_\eta(L) = \mathbf{E}_{1-\eta}(L)$  and in particular  $\mathbf{E}_0(L) = \mathbf{E}_1(L) = L$ . Thus, the case  $\eta = 1/2$  is special.

**Definition 2.4.** The set  $\mathbf{E}_{1/2}(L)$  is the **Wigner caustic** of  $L$ .

Consider  $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$  with coordinates  $(x^+, x^-)$  and the tangent bundle to  $\mathbb{R}^{2m}$ ,  $T\mathbb{R}^{2m} = \mathbb{R}^{2m} \times \mathbb{R}^{2m}$ , with coordinates  $(x, \dot{x})$  and standard projection  $\pi : T\mathbb{R}^{2m} \ni (x, \dot{x}) \rightarrow x \in \mathbb{R}^{2m}$ . Consider the linear map

$$\Phi_{1/2} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \ni (x^+, x^-) \mapsto \left( \frac{x^+ + x^-}{2}, \frac{x^+ - x^-}{2} \right) = (x, \dot{x}) \in T\mathbb{R}^{2m}.$$

On the product affine symplectic space, consider the symplectic form

$$\delta_{1/2}\omega = \frac{1}{2}(\pi_1^*\omega - \pi_2^*\omega) ,$$

$\pi_i$  the  $i$ -th projection  $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ . Canonical relations correspond to Lagrangian submanifolds of  $(\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$ . Then,

$$\left( \Phi_{1/2}^{-1} \right)^* (\delta_{1/2}\omega) = \dot{\omega} ,$$

where  $\dot{\omega}$  is the canonical symplectic form on  $T\mathbb{R}^{2m}$ , which is defined by  $\dot{\omega}(x, \dot{x}) = d\{\dot{x} \lrcorner \omega\}(x)$  or, in Darboux coordinates for  $\omega$ , by

$$\dot{\omega} = \sum_{i=1}^m d\dot{p}_i \wedge d\dot{q}_i + d\dot{p}_i \wedge d\dot{q}_i .$$

If  $L$  is a Lagrangian submanifold of  $(\mathbb{R}^{2m}, \omega)$ , then  $L \times L$  is a Lagrangian submanifold of  $(\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$  and  $\mathcal{L} = \Phi_{1/2}(L \times L)$  is a Lagrangian submanifold of  $(T\mathbb{R}^{2m}, \dot{\omega})$ , which can be locally described by a generating function of the midpoints  $x = \pi \circ \Phi_{1/2}(x^+, x^-)$ ,  $(x^+, x^-) \in L \times L$ , when  $\mathcal{L}$  projects regularly to the zero section [14][16].

We recall basic definitions of the theory of Lagrangian singularities (see [1], [7]). First,  $(T\mathbb{R}^{2m}, \dot{\omega})$  with canonical projection  $\pi : T\mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$  is a *Lagrangian fibre bundle* and  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{R}^{2m}$  is a *Lagrangian map*. Let  $\tilde{\mathcal{L}}$  be another Lagrangian submanifold of  $(T\mathbb{R}^{2m}, \dot{\omega})$ . Two Lagrangian maps  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{R}^{2m}$  and  $\pi|_{\tilde{\mathcal{L}}} : \tilde{\mathcal{L}} \rightarrow \mathbb{R}^{2m}$  are *Lagrangian equivalent* if there exists a symplectomorphism of  $(T\mathbb{R}^{2m}, \dot{\omega})$  taking fibres of  $\pi$  to fibres and mapping  $\mathcal{L}$  to  $\tilde{\mathcal{L}}$ . A Lagrangian map is *stable* if every nearby Lagrangian map (in the Whitney topology) is Lagrangian equivalent to it. The set of critical values of a Lagrangian map is called a *caustic*. Then, we have the following result:

**Proposition 2.5** ([7]). *The caustic of the Lagrangian map  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{R}^{2m}$  is the Wigner caustic  $\mathbf{E}_{1/2}(L)$ .*

In this paper, we study  $\mathbf{E}_{1/2}(L)$  in a neighborhood  $L$ . For this reason, we consider pairs of points of the type  $(a, a) \in L \times L$  as strongly parallel pairs. In other words, in Definition 2.1 *we did not impose the restriction  $a \neq b$  on the pair of points of  $L$  to be considered a parallel pair*. This broader definition of parallel pairs is suitable for studying the part of the Wigner caustic that is close to  $L$ , because then  $L$  is itself part of the Wigner caustic. This broader definition of the Wigner caustic is also natural from its origin in quantum physics, as shown by equations (2.3) and (2.6). On the other hand, imposing the restriction  $a \neq b$  in Definition 2.1 allows for a neater definition of the Wigner caustic as a centre symmetry set, as in [7] (see also [9], where, for a curve  $L$  and  $a \neq b$ ,  $\mathbf{E}_{1/2}(L)$  is called the area evolute of  $L$ ).

**Definition 2.6.** The germ at  $a$  of the **Wigner caustic on shell** is the germ of Wigner caustic  $\mathbf{E}_{1/2}(L)$  at the point  $a \in L$ .

Now let  $L$  be a germ at 0 of a smooth Lagrangian submanifold of  $(\mathbb{R}^{2m}, \omega)$ , generated by the function-germ  $S \in \mathcal{E}_m$  in the usual way,

$$(2.7) \quad L = \left\{ (p, q) \in \mathbb{R}^{2m} : p_i = \frac{\partial S}{\partial q_i}(q) \text{ for } i = 1, \dots, m \right\}.$$

Then,  $\mathcal{L}$  is the germ at 0 of a submanifold of  $(T\mathbb{R}^{2m}, \dot{\omega})$  described as

$$(2.8) \quad \dot{p} = \frac{1}{2} \left( \frac{\partial S}{\partial q}(q + \dot{q}) - \frac{\partial S}{\partial q}(q - \dot{q}) \right),$$



$$(2.9) \quad p = \frac{1}{2} \left( \frac{\partial S}{\partial q}(q + \dot{q}) + \frac{\partial S}{\partial q}(q - \dot{q}) \right).$$

By Proposition 2.5, the germ at  $0 \in L$  of the Wigner caustic on shell  $\mathbf{E}_{1/2}(L)$  is described as

$$(2.10) \quad \begin{aligned} & \exists \dot{q} \in \mathbb{R}^m \text{ s.t. (2.9) is satisfied, and} \\ & \det \left[ \frac{\partial^2 S}{\partial q^2}(q + \dot{q}) - \frac{\partial^2 S}{\partial q^2}(q - \dot{q}) \right] = 0. \end{aligned}$$

Thus, putting  $\dot{q} = 0$  in (2.9)-(2.10) we obtain the obvious fact:

**Proposition 2.7.**  *$L$  is contained in  $\mathbf{E}_{1/2}(L)$ .*

Now, let us consider the reflection

$$(2.11) \quad \zeta : T\mathbb{R}^{2m} \ni (\dot{p}, \dot{q}, p, q) \mapsto (-\dot{p}, -\dot{q}, p, q) \in T\mathbb{R}^{2m}$$

whose mirror is the zero section  $\{\dot{p} = \dot{q} = 0\} \subset T\mathbb{R}^{2m}$ . Note that  $\{id, \zeta\}$  generates an action of  $\mathbb{Z}_2$  on  $T\mathbb{R}^{2m}$ . Using (2.8) we obtain

**Proposition 2.8.**  *$\mathcal{L}$  is  $\mathbb{Z}_2$ -symmetric, that is,  $\zeta(\mathcal{L}) = \mathcal{L}$ .*

We shall study singularities of  $\mathbf{E}_{1/2}(L)$  via generating families of  $\mathcal{L}$ .

**Definition 2.9.** The germ of a **generating family** of  $\mathcal{L}$  is the smooth function-germ  $F : \mathbb{R}^k \times \mathbb{R}^{2m} \ni (\beta, p, q) \mapsto F(\beta, p, q) \in \mathbb{R}$  such that

$$(2.12) \quad \mathcal{L} = \left\{ (\dot{p}, \dot{q}, p, q) \in T\mathbb{R}^{2m} : \exists \beta \in \mathbb{R}^k \quad \dot{p} = \frac{\partial F}{\partial q}, \quad \dot{q} = -\frac{\partial F}{\partial p}, \quad \frac{\partial F}{\partial \beta} = 0 \right\}.$$

**Remark 2.10.** When there are no symmetries, two Lagrangian map-germs on the same Lagrangian fibre bundle are Lagrangian equivalent if and only if their generating families are stably (fibred)  $\mathcal{R}^+$ -equivalent. Moreover the Lagrangian map-germ given by the generating family  $F(\beta, p, q)$  with parameters  $(p, q)$  is Lagrangian stable if and only if  $F(\beta, p, q)$  is a  $\mathcal{R}^+$ -versal deformation of  $f(\beta) = F(\beta, 0, 0)$  (see [1]).

Now, in the  $\mathbb{Z}_2$ -symmetric context, the following Theorem, whose proof is a straightforward computation from (2.12) to (2.8)-(2.9), is a particular case of the more general result presented in [7]:

**Theorem 2.11** ([7]). *The germ at  $0 \in L$  of the Wigner caustic on shell is the germ of a caustic of the germ of a Lagrangian submanifold  $\mathcal{L}$  in the Lagrangian fibre bundle  $T\mathbb{R}^{2m} \ni (\dot{p}, \dot{q}, p, q) \mapsto (p, q) \in \mathbb{R}^{2m}$  with the symplectic form  $\dot{\omega} = \sum_{i=1}^m d\dot{p}_i \wedge dq_i + dp_i \wedge d\dot{q}_i$  and generating family*

$$(2.13) \quad F(\beta, p, q) \equiv \frac{1}{2}S(q + \beta) - \frac{1}{2}S(q - \beta) - \sum_{i=1}^m p_i \beta_i.$$

For any  $\beta, p, q$ , the generating family (2.13) satisfies

$$(2.14) \quad F(-\beta, p, q) \equiv -F(\beta, p, q)$$

It implies that  $F$  is a deformation of an *odd function-germ*

$$(2.15) \quad f(\beta) \equiv F(\beta, 0, 0) \equiv \frac{1}{2}(S(\beta) - S(-\beta)).$$

We call  $F$  which satisfies (2.14) an *odd deformation* of an odd function-germ  $f$  (see Definitions 3.1 and 3.7, below). Thus, in order to study singularities of the Wigner caustic on shell, we must consider classification of odd function-germs and their odd deformations.

**Remark 2.12.** Theorem 2.11 implies that singularities of the Wigner caustic on shell are  $\mathbb{Z}_2$ -symmetric singularities (see Proposition 2.8, above, and Remark 3.3, below). However, at the level of a germ of the Wigner caustic on shell  $\mathbf{E}_{1/2}(L) \subset \mathbb{R}^{2m}$ , this  $\mathbb{Z}_2$ -symmetry is a *hidden symmetry* which is only actually revealed in  $\mathcal{L} \subset T\mathbb{R}^{2m}$ .

**Remark 2.13.** The form (2.13) for the generating family of the Wigner caustic on shell of a Lagrangian submanifold of the affine-symplectic space was already presented in [13], and its odd character was remarked. However, the classification used there, borrowed from Arnold's, was not performed in the  $\mathbb{Z}_2$ -symmetric context. Furthermore, albeit respecting that  $f(\beta) = F(\beta, 0, 0)$  is odd, the authors did not take into account that  $F(\beta, p, q)$  must be an odd deformation of  $F(\beta, 0, 0)$ .

### 3. SINGULARITIES OF ODD FUNCTIONS

**3.1. Preliminaries.** We recall basic definitions.

**Definition 3.1.** A smooth function-germ  $f$  at 0 on  $\mathbb{R}^m$  is **even** if  $f(-x) \equiv f(x)$  and it is **odd** if  $f(-x) \equiv -f(x)$ .

**Notation 3.2.** Let us denote by  $\mathcal{E}_m^{\text{even}}$  the ring of even smooth function-germs  $f : (\mathbb{R}^m, 0) \rightarrow \mathbb{R}$  and by  $\mathcal{E}_m^{\text{odd}}$  the set of odd smooth function-germs  $g : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}, 0)$ , which has a module structure over  $\mathcal{E}_m^{\text{even}}$ .

**Remark 3.3.** Consider the diagonal action of  $\mathbb{Z}_2 = \{1, -1\}$  on  $\mathcal{R}^m$ :

$$(3.1) \quad \begin{array}{ccc} \mathbb{Z}_2 \times \mathbb{R}^m & \rightarrow & \mathbb{R}^m \\ (\gamma, (x_1, \dots, x_m)) & \mapsto & (\gamma x_1, \dots, \gamma x_m). \end{array}$$

Hence,  $\mathcal{E}_m^{\text{even}}$  is the ring of  $\mathbb{Z}_2$ -invariant germs under this action on source. Also,  $\mathcal{E}_m^{\text{odd}}$  is the module of  $\mathbb{Z}_2$ -equivariant germs, with same action on source and on target - take (3.1) for  $m = 1$ .

We now set up the equivalence relation in  $\mathcal{E}_m^{\text{odd}}$ . Changes of coordinates shall preserve  $\mathbb{Z}_2$ -equivariance, so we consider the following:

**Definition 3.4.** A diffeomorphism-germ  $\Phi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$  is **odd** if  $\Phi(-x) \equiv -\Phi(x)$ . Denote by  $\mathcal{D}_m^{odd}$  the group of odd diffeomorphism-germs  $(\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$ .

**Definition 3.5.** Let  $f, g \in \mathcal{E}_m^{odd}$ . We say that  $f$  and  $g$  are  $\mathcal{R}^{odd}$ -**equivalent** if there exists  $\Phi \in \mathcal{D}_m^{odd}$  such that  $f = g \circ \Phi$ .

Following standard notation, denote by  $L\mathcal{R}^{odd}g$  the tangent space to the  $\mathcal{R}^{odd}$ -orbit of  $g$  at  $g$ , given by elements of the form  $\frac{d}{dt}|_{t=0} (g \circ \Phi^t) = \sum_{i=1}^m \frac{\partial g}{\partial x_i} \frac{d\phi_i^t}{dt}|_{t=0}$ , where  $g \circ \Phi^t$  is a path in the  $\mathcal{R}^{odd}$ -orbit of  $g$ , with  $\Phi^t = (\phi_1^t, \dots, \phi_m^t)$  in  $\mathcal{D}_m^{odd}$  such that  $\Phi^0 = I$ . Now,  $\phi_i^t = \sum_{j=1}^m x_j h_{ij}^t$ , with  $h_{ij}^t \in \mathcal{E}_m^{even}$ , so that  $\frac{d}{dt}|_{t=0} (g \circ \Phi^t) = \sum_{i,j=1}^m x_j \frac{\partial g}{\partial x_i} \frac{dh_{ij}^t}{dt}|_{t=0}$ ,  $i, j = 1, \dots, m$ . Since  $h_{ij}^t \in \mathcal{E}_m^{even}$ , so does  $\frac{dh_{ij}^t}{dt}|_{t=0}$ . We have:

**Proposition 3.6.** Let  $g \in \mathcal{E}_m^{odd}$ . The tangent space  $L\mathcal{R}^{odd}g$  to the  $\mathcal{R}^{odd}$ -orbit of  $g$  at  $g$  is the  $\mathcal{E}_m^{even}$ -module generated by  $\left\{ x_j \frac{\partial g}{\partial x_i} : i, j = 1, \dots, m \right\}$ .

**Definition 3.7.** A function-germ  $F \in \mathcal{E}_{m+k}$  is an **odd deformation** of  $f \in \mathcal{E}_m^{odd}$  if  $F|_{\mathbb{R}^m \times \{0\}} = f$  and for any fixed  $\lambda \in \mathbb{R}^k$  the function-germ  $F|_{\mathbb{R}^m \times \{\lambda\}} \in \mathcal{E}_m^{odd}$ . The space  $\mathbb{R}^k$  is called the **base** of the odd deformation  $F$  and  $k$  is its **dimension**.

**Definition 3.8.** The odd deformation  $F \in \mathcal{E}_{m+k}$  is  $\mathcal{R}^{odd}$ -**versal** if every odd deformation of  $f$  is  $\mathcal{R}^{odd}$ -isomorphic to one induced from  $F$  i.e. any odd deformation  $G \in \mathcal{E}_{m+l}$  of  $f$  is representable in the form

$$G(x, \lambda) \equiv F(\Phi(x, \lambda), \Lambda(\lambda)),$$

$\Phi : (\mathbb{R}^{m+k}, 0) \rightarrow (\mathbb{R}^m, 0)$ ,  $\Lambda : (\mathbb{R}^k, 0) \rightarrow (\mathbb{R}^l, 0)$  smooth map-germs s.t.

$$\Phi|_{\mathbb{R}^m \times \{\lambda\}} \in \mathcal{D}_m^{odd}, \quad \Phi(x, 0) \equiv x.$$

An  $\mathcal{R}^{odd}$ -versal deformation  $F \in \mathcal{E}_{m+k}$  of  $f \in \mathcal{E}_m^{odd}$  is  $\mathcal{R}^{odd}$ -**miniversal** if the dimension of the base has its least possible value. This minimum value being the (odd) codimension of  $f$ .

The group  $\mathcal{D}_m^{odd}$  is a geometric subgroup in the sense of Damon (see [6]). The following theorem is a particular case of [2, Theorem 3.7]:

**Theorem 3.9.** Let  $g \in \mathcal{E}_m^{odd}$ . Then

(a) A  $k$ -parameter deformation  $G$  of  $g$  is  $\mathcal{R}^{odd}$ -versal if and only if

$$\mathcal{E}_m^{odd} = \mathcal{E}_m^{even} \left\{ x_j \frac{\partial g}{\partial x_i} : i, j = 1, \dots, m \right\} + \mathbb{R} \left\{ \frac{\partial G}{\partial \lambda_\ell} |_{\mathbb{R}^m \times \{0\}} : \ell = 1, \dots, k \right\}.$$

(b) If  $W \subset \mathcal{E}_m^{odd}$  is a finite dimensional vector space such that  $\mathcal{E}_m^{odd} = L\mathcal{R}^{odd}g \oplus W$ , and if  $h_1, \dots, h_s \in \mathcal{E}_m^{odd}$  is a basis for  $W$ , then  $G(x, \lambda) \equiv g(x) + \sum_{j=1}^s \lambda_j h_j(x)$  is a  $\mathcal{R}^{odd}$ -miniversal deformation of  $g$ .

We introduce the equivalence relation between odd deformations.

**Definition 3.10.** Odd deformations  $F, G \in \mathcal{E}_{m+k}$  are **fibred  $\mathcal{R}^{odd}$ -equivalent** if there exists a fibred diffeomorphism-germ  $\Psi \in \mathcal{D}_{m+k}$  s.t.  $\Psi(x, \lambda) \equiv (\Phi(x, \lambda), \Lambda(\lambda))$ ,  $\Phi|_{\mathbb{R}^m \times \{\lambda\}} \in \mathcal{D}_m^{odd}$ ,  $\forall \lambda \in \mathbb{R}^k$ , and  $F = G \circ \Psi$ .

**Notation 3.11.** Let  $\mathcal{M}_m^{k(odd)}$  denote the  $\mathcal{E}_m^{even}$ -submodule of  $\mathcal{E}_m^{odd}$  generated by  $x_1^{k_1} \cdots x_m^{k_m}$ ,  $\forall k_1, \dots, k_m \geq 0$ , s.t.  $k_1 + \dots + k_m = k$ .

Obviously, these are nontrivial submodules precisely when  $k$  is odd. It follows the finite determinacy result for our particular case (see [6], [17]-[18]):

**Proposition 3.12.**  $g \in \mathcal{E}_m^{odd}$  is finitely  $\mathcal{R}^{odd}$ -determined if and only if  $\mathcal{M}_m^{k(odd)} \subset L\mathcal{R}^{odd}g$  for some odd positive integer  $k$ .

**Theorem 3.13.** Let  $g \in \mathcal{E}_m^{odd}$  with a singular point at 0. If  $m \geq 3$ , then  $g$  is not  $\mathcal{R}^{odd}$ -simple.

*Proof.* If 0 is a singular point of  $g$  then  $g \in \mathcal{M}_m^{3(odd)}$ . Dimension of the space of 3-jets at 0 of singular odd function-germs is  $(m+2)(m+1)m/6$ . We act on this space with  $GL(m)$ , of dimension  $m^2$ . But, for  $m \geq 3$ ,  $(m+2)(m+1)m/6 > m^2$ .  $\square$

**Remark 3.14.** If  $g \in \mathcal{E}_{2+n}^{odd}$ , the usual procedure of adding quadratic forms in the remaining  $n$  variables cannot be performed.

Thus, classification of simple odd singularities must be performed only in dimension one and two, as presented in the next subsection.

### 3.2. Simple odd function-germs and their odd deformations.

Here we deduce the normal forms and their mini-versal deformations for the simple odd singularities of function germs in one and two variables. We have chosen a particular notation for each. We start with the cases in one-variable. The results are obtained straightforwardly and are given in the next theorem and corollary. The following theorem and corollary deal with the cases in two variables.

**Theorem 3.15.** Let  $g \in \mathcal{E}_1^{odd}$ . Then  $g$  is  $\mathcal{R}^{odd}$ -simple if, and only if,  $g$  is  $\mathcal{R}^{odd}$ -equivalent to one of the following function-germs at 0:

$$A_{2k/2} : x \mapsto x^{2k+1}, \text{ for } k = 1, 2, \dots$$

**Corollary 3.16.** For  $k \geq 1$ ,  $\mathcal{R}^{odd}$ -miniversal deformation of  $A_{2k/2}$  is

$$G(x, \lambda_1, \dots, \lambda_k) = x^{2k+1} + \sum_{j=1}^k \lambda_j x^{2j-1}.$$

**Theorem 3.17.** *Let  $g \in \mathcal{E}_2^{odd}$ . Then  $g$  is  $\mathcal{R}^{odd}$ -simple if, and only if,  $g$  is  $\mathcal{R}^{odd}$ -equivalent to one of the following function-germs at 0:*

$$D_{2k/2}^{\pm} : (x_1, x_2) \mapsto x_1^2 x_2 \pm x_2^{2k-1}, \text{ for } k = 2, 3, \dots$$

$$E_{8/2} : (x_1, x_2) \mapsto x_1^3 + x_2^5,$$

$$J_{10/2}^{\pm} : (x_1, x_2) \mapsto x_1^3 \pm x_1 x_2^4.$$

$$E_{12/2} : (x_1, x_2) \mapsto x_1^3 + x_2^7.$$

*Proof.* The procedure is the systematic usage of the complete transversal method ([5], [11]) at the level of jets and then usage of the finite determinacy theorem. In our context, the complete transversal is a subspace  $T$  of  $\mathcal{M}_m^{2k+1(odd)}$  such that

$$(3.2) \quad \mathcal{M}_2^{2k+1(odd)} \subset L\mathcal{R}_1^{odd} \cdot g + T + \mathcal{M}_2^{2k+3(odd)},$$

where  $\mathcal{R}_1^{odd}$  is the subgroup of  $\mathcal{R}^{odd}$  whose elements have 1-jet equal to identity, and  $L\mathcal{R}_1^{odd} \cdot g$  is the tangent space to the  $\mathcal{R}_1^{odd}$ -orbit of  $g$  at  $g$ .

We start with the 3-jet of  $g$ , which is also the starting point of the classification without symmetry. Since linear changes of coordinates are  $\mathbb{Z}_2$ -equivariant, it follows that, at this level, the results here are precisely the same as in the context without symmetry. Therefore, as it is well known, a nonzero cubic polynomial in two variables is linearly equivalent to one of the following types:

$$(3.3) \quad x_1^2 x_2 \pm x_2^3$$

$$(3.4) \quad x_1^2 x_2$$

$$(3.5) \quad x_1^3$$

First, assuming that  $j_0^3 g$  is of form (3.3), the  $\mathcal{R}_1^{odd}$  tangent space of the orbit of (3.3) is  $\mathcal{M}_2^{5(odd)}$ . The complete transversal is empty in this case and  $g$  is finitely  $\mathcal{R}^{odd}$ -determined and  $\mathcal{R}^{odd}$ -equivalent to (3.3).

Now, assume that  $j_0^3 g$  has form (3.4), whose orbit has

$$\mathcal{E}_2^{even} \cdot \{x_1^5, x_1^4 x_2, x_1^3 x_2^2, x_1^2 x_2^3, x_1 x_2^4\}$$

as its  $\mathcal{R}_1^{odd}$  tangent space. So the complete transversal is  $T = \mathbb{R}\{x_2^5\}$ . Hence,  $j_0^5 g$  is  $\mathcal{R}_1^{odd}$ -equivalent to  $x_1^2 x_2 + a x_2^5$  and it is easy to see that if  $a > 0$  then  $j_0^5 f$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^2 x_2 + x_2^5$ , and if  $a < 0$  then  $j_0^5 f$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^2 x_2 - x_2^5$ . In the next step we check that the  $\mathcal{R}_1^{odd}$  tangent space to the orbit of both of these germs is  $\mathcal{M}_2^{5(odd)}$ . So the complete transversal is empty and  $g$  is finitely  $\mathcal{R}^{odd}$ -determined and  $\mathcal{R}^{odd}$ -equivalent to  $x_1^2 x_2 \pm x_2^5$ . If  $a = 0$ , then  $T = \mathbb{R}\{x_2^7\}$  and  $j_0^7 g$  is  $\mathcal{R}_1^{odd}$ -equivalent to  $x_1^2 x_2 + b x_2^7$ . Proceeding inductively, we obtain that

if  $j^3g_0$  has the form (3.4) and  $g$  is finitely  $\mathcal{R}^{odd}$ -determined then  $g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^2x_2 \pm x_2^{2k+1}$  for  $k \geq 2$ .

Finally, assume that  $j_0^3g$  has the form (3.5). In this case,  $T = \mathbb{R}\{x_1x_2^4, x_2^5\}$  and  $j_0^5g$  is  $\mathcal{R}_1^{odd}$ -equivalent to  $x_1^3 + ax_2^5 + bx_1x_2^4$ .

If  $a \neq 0$ , then  $j_0^5g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 + x_2^5 + bx_1x_2^4$  and

$$\mathcal{E}_2^{even} \cdot \{x_2^5, x_1x_2^4, x_1^2x_2, x_1^3\}$$

is its tangent space. So its dimension does not depend on  $b$  and it contains the germ of  $x_1x_2^4$ . It then follows from Mather's lemma that  $j_0^5g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 + x_2^5$ . As next step we obtain that  $g$  is finitely  $\mathcal{R}^{odd}$ -determined. Then,  $g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 + x_2^5$ .

If  $a = 0$  and  $b \neq 0$ , then  $j_0^5g = x_1^3 \pm x_1x_2^4$  and  $T = \mathbb{R}\{x_2^7\}$ . Then  $j_0^7g$  is  $\mathcal{R}_1^{odd}$ -equivalent to  $x_1^3 \pm x_1x_2^4 + ax_2^7$ . But  $L\mathcal{R}^{odd}g$  is given by

$$\mathcal{E}_2^{even} \cdot \{x_2^7, x_1x_2^4, 3x_1^2x_2 \pm x_2^5, x_1^3\}.$$

Its dimension indepdends on  $a$  and it contains  $x_2^7$ . By Mather's lemma,  $j_0^7g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 \pm x_1x_2^4$ . As in the previous case, we find that  $g$  is finitely  $\mathcal{R}^{odd}$ -determined, so is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 \pm x_1x_2^4$ .

If  $a = b = 0$ , then  $j_0^5g = x_1^3$ . Thus, the complete transversal is  $T = \mathbb{R}\{x_1x_2^6, x_2^7\}$ . It means that  $j_0^7g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 + cx_1x_2^6 + dx_2^7$ . If  $d \neq 0$  we may assume that  $j_0^7g = x_1^3 + cx_1x_2^6 + x_2^7$ . But  $L\mathcal{R}^{odd}g$  is

$$\mathcal{E}_2^{even} \cdot \{x_2^7, x_1x_2^6, x_1^2x_2, x_1^3\}.$$

Its dimension indepdends on  $c$  and it contains  $x_1x_2^6$ . By Mather's lemma,  $j_0^7g$  is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 + x_2^7$ . As next step we obtain that  $g$  is finitely  $\mathcal{R}^{odd}$ -determined, so is  $\mathcal{R}^{odd}$ -equivalent to  $x_1^3 + x_2^7$ .

If  $d = 0$  and  $c \neq 0$ , we may assume  $j_0^7g = x_1^3 \pm x_1x_2^6$ . The complete transversal is  $T = \mathbb{R}\{x_2^9\}$ . So  $j_0^9g$  is  $\mathcal{R}_1^{odd}$ -equivalent to  $x_1^3 \pm x_1x_2^6 + ax_2^9$ . But  $x_2^9 \notin L\mathcal{R}^{odd}j_0^9g$ . By Mather's lemma,  $c$  is a modulus.  $\square$

From Theorem 3.9 and the proof of Theorem 3.17, we obtain:

**Corollary 3.18.** *The  $\mathcal{R}^{odd}$ -miniversal deformation of the odd-simple map-germs are given by:*

$$D_{2k/2}^\pm : F(x_1, x_2, \lambda_1, \dots, \lambda_k) \equiv x_1^2x_2 \pm x_2^{2k-1} + \lambda_1x_1 + \sum_{i=2}^k \lambda_i x_2^{2i-3}.$$

$$E_{8/2} : F(x_1, x_2, \lambda_1, \dots, \lambda_4) \equiv x_1^3 + x_2^5 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_1x_2^2 + \lambda_4x_2^3.$$

$$J_{10/2}^\pm : F(x_1, x_2, \lambda_1, \dots, \lambda_5) \equiv$$

$$x_1^3 \pm x_1x_2^4 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_1^2x_2 + \lambda_4x_2^2x_1 + \lambda_5x_2^3.$$

$$E_{12/2} : F(x_1, x_2, \lambda_1, \dots, \lambda_6) \equiv$$

$$x_1^3 + x_2^7 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_1x_2^2 + \lambda_4x_2^3 + \lambda_5x_1x_2^5 + \lambda_6x_2^6.$$

**Remark 3.19.** The notations for the odd-simple singularities presented above have been chosen by their resemblance with the classical notations [1] for normal forms of  $\mathcal{R}$ -singularities. In fact,  $A_{2k/2}$  has the same representative as  $A_{2k}$ , but while the latter has codimension  $2k$ , the former has odd codimension  $k = 2k/2$ . Similarly, for  $D_{2k/2}$  and  $E_{8/2}$ , with odd codimensions  $k$  and  $4$ , respectively, for which the corresponding  $\mathcal{R}$ -singularities  $D_{2k}$  and  $E_8$  have codimensions  $2k$  and  $8$ , respectively. The situation differs for the other odd-simple singularities. The germ of the odd codimension 6 singularity  $E_{12/2}$  is  $\mathcal{R}$ -equivalent to the codimension 12 singularity  $E_{12}$ , but we stress that the latter is unimodal. Similarly for the odd codimension 5 singularity  $J_{10/2}^\pm$  in comparison with codimension 10 unimodal  $\mathcal{R}$ -singularity  $J_{10}$ .

#### 4. SIMPLE STABLE SINGULARITIES OF WIGNER CAUSTIC ON SHELL

From classical results ([1]) we know that Lagrangian equivalence of Lagrangian maps corresponds to stable fibred  $\mathcal{R}^+$ -equivalence of their generating families (see Remark 2.10). Thus we introduce the following definition in the  $\mathbb{Z}_2$ -symmetric case.

**Definition 4.1.** Let  $L$  and  $\tilde{L}$  be germs at  $(0, 0) \in \mathbb{R}^{2m}$  of Lagrangian submanifolds of the affine symplectic space. The germs at  $(0, 0)$  of Wigner caustics on shell  $\mathbf{E}_{1/2}(L)$  and  $\mathbf{E}_{1/2}(\tilde{L})$  are **Lagrangian equivalent** if germs at  $(0, 0, 0) \in \mathbb{R}^m \times \mathbb{R}^{2m}$  of the corresponding odd generating families  $F$  and  $\tilde{F}$  are fibred  $\mathcal{R}^{odd}$ -equivalent.

From Remark 2.12, this means equivalence of  $\mathbb{Z}_2$ -symmetric germs of Wigner caustics. The following definition specializes to this  $\mathbb{Z}_2$ -symmetric context the well-known fact ([1]) that stability of Lagrangian maps corresponds to versality of generating families (Remark 2.10).

**Definition 4.2.** A germ of Wigner caustic on shell is **stable** if its generating family is an  $\mathcal{R}^{odd}$ -versal deformation of an odd function-germ, and it is **simple stable** if its generating family is an  $\mathcal{R}^{odd}$ -versal deformation of an  $\mathcal{R}^{odd}$ -odd simple function-germ.

Notice that any odd function-germ  $f \in \mathcal{M}_m^{3(odd)}$  can be written as  $f(\beta) \equiv \frac{1}{2}(S(\beta) - S(-\beta))$  for some  $S \in \mathcal{M}_m^3$ , implying the following:

**Proposition 4.3.** *For any  $f \in \mathcal{M}_m^{3(odd)}$  there exists  $S \in \mathcal{M}_m^3$  such that the generating family  $F$  of the form (2.13) is an odd deformation of  $f$ .*

By Theorem 3.9 we obtain the following corollary.

**Corollary 4.4.** *The germ of a generating family  $F$  of the form (2.13) is an  $\mathcal{R}^{odd}$ -versal deformation if and only if*

$$(4.1) \mathcal{M}_m^{3(odd)} = \mathcal{E}_m^{even} \left\{ \beta_i \left( \frac{\partial S}{\partial q_j}(\beta) + \frac{\partial S}{\partial q_j}(-\beta) \right) : i, j = 1, \dots, m \right\} + \mathbb{R} \left\{ \frac{\partial S}{\partial q_j}(\beta) - \frac{\partial S}{\partial q_j}(-\beta) : j = 1, \dots, m \right\}.$$

From Corollary 4.4 we get the following realization theorem.

**Theorem 4.5.** *Let  $f \in \mathcal{M}_m^{3(odd)}$  be a finitely determined germ. Then there exists  $S \in \mathcal{M}_m^3$  such that the generating family  $F$  of the form (2.13) is an  $\mathcal{R}^{odd}$ -versal deformation of  $f$  if and only if there exist  $h_1, \dots, h_m$  in  $\mathcal{M}_m^{3(odd)}$  such that*

$$(4.2) \quad \mathcal{M}_m^{3(odd)} = L\mathcal{R}^{odd}f + \mathbb{R}\{h_1, \dots, h_m\}$$

and  $\sum_{i=1}^m h_i(\beta_1, \dots, \beta_m)d\beta_i$  is a germ of closed 1-form.

*Proof.* First, notice that any function-germ  $S \in \mathcal{E}_m$  can be decomposed into  $S = S^+ + S^-$ , where  $S^+ \in \mathcal{E}_m^{even}$ ,  $S^- \in \mathcal{E}_m^{odd}$  are given in the following way  $S^+(\beta) \equiv \frac{1}{2}(S(\beta) + S(-\beta))$ ,  $S^-(\beta) \equiv \frac{1}{2}(S(\beta) - S(-\beta))$ . Then the versality condition (4.1) of  $F$  given by (2.13) has the form

$$\mathcal{M}_m^{3(odd)} = L\mathcal{R}^{odd}S^- + \mathbb{R} \left\{ \frac{\partial S^+}{\partial \beta_j}(\beta) : j = 1, \dots, m \right\}.$$

From the above,  $f$  must be equal to  $S^-$  and the germ of a 1-form  $\sum_{j=1}^m \frac{\partial S^+}{\partial \beta_j}(\beta)d\beta_j$  is closed since it is just  $dS^+$ . On the other hand if condition (4.2) is satisfied and  $\alpha = \sum_{i=1}^m h_i(\beta_1, \dots, \beta_m)d\beta_i$  is a germ of closed 1-form then it is obvious that there exists such a function-germ  $g \in \mathcal{E}_m^{even}$  such that  $\alpha = dg$ . So we take  $S = f + g$ .  $\square$

It follows from Theorem 3.13 that simple singularities for the Wigner caustic on shell of a Lagrangian submanifold can be realized only for curves in  $\mathbb{R}^2$  and surfaces in  $\mathbb{R}^4$ . Thus, first we apply Theorem 4.5 to check which versal deformations of simple odd singularities are realizable as a generating family of the form (2.13).

**Corollary 4.6.**  *$\mathcal{R}^{odd}$ -versal deformations of  $A_{2/2}$ ,  $A_{4/2}$  (for  $m = 1$ ) and  $D_{4/2}^\pm$ ,  $D_{6/2}^\pm$ ,  $D_{8/2}^\pm$ ,  $E_{8/2}$  (for  $m = 2$ ) are realizable as generating families of form (2.13).*

*$\mathcal{R}^{odd}$ -versal deformations of  $A_{2k/2}$  for  $k > 2$  (and for  $m = 1$ ) and  $D_{2k/2}$  for  $k > 4$ ,  $J_{10/2}^\pm$  and  $E_{12/2}$  (for  $m = 2$ ) are not realizable as generating families of form (2.13).*



*Proof.* First notice that if the codimension of the singularity is greater than  $2m$  then the  $\mathcal{R}^{odd}$ -versal deformation of it is not realizable by a generating family of the form (2.13). This proves the second statement. Since any smooth 1-form on  $\mathbb{R}$  is closed this is the only restriction for  $m = 1$ . The realization of  $D_{4/2}^\pm$  is obvious. For the others singularities we apply Theorem 4.5 in the following way: for  $D_{6/2}^\pm$  take  $h_1(\beta) \equiv 0$  and  $h_2(\beta) \equiv \beta_2^3$ , for  $D_{8/2}^\pm$  take  $h_1(\beta) \equiv \beta_2^3$  and  $h_2(\beta) \equiv \beta_2^5 + 3\beta_1\beta_2^2$ , and for  $E_{8/2}$  take  $h_1(\beta) \equiv \beta_2^3$  and  $h_2(\beta) \equiv 3\beta_1\beta_2^2$ .  $\square$

**4.1. The Wigner caustic on shell of a Lagrangian curve.** Let  $L$  be the germ at  $(0,0)$  of a curve on symplectic affine plane  $(\mathbb{R}^2, \omega = dp \wedge dq)$  and, without loss of generality, assume that  $L$  is generated by a function-germ  $S \in \mathcal{M}_1^3 \subset \mathcal{E}_1$  in the usual way given by (2.7),  $i = 1$ .

**Theorem 4.7.** *Let  $F$  of form (2.13) be the generating family of  $\mathcal{L}$ .*

*If  $\frac{d^3 S}{dq^3}(0) \neq 0$ ,  $F$  is fibred  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $A_{2/2} : (\beta, p, q) \mapsto \beta^3 + p\beta$ .*

*If  $\frac{d^3 S}{dq^3}(0) = 0$ ,  $\frac{d^4 S}{dq^4}(0) \neq 0$  and  $\frac{d^5 S}{dq^5}(0) \neq 0$ ,  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $A_{4/2} : (\beta, p, q) \mapsto \beta^5 + q\beta^3 + p\beta$ .*

*Proof.* From (2.13),  $\frac{\partial^k F}{\partial \beta^k}(\beta, 0, 0) = \frac{1}{2} \left( \frac{d^k S}{dq^k}(\beta) + (-1)^{k+1} \frac{d^k S}{dq^k}(-\beta) \right)$ . Thus  $\frac{d^3 S}{dq^3}(0) \neq 0$  implies that  $\frac{d^3 F}{d\beta^3}(0, 0, 0) \neq 0$  and  $F|_{\mathbb{R} \times \{0\} \times \{0\}} \in \mathcal{M}_1^3$ , since  $S \in \mathcal{M}_1^3$ . Therefore,  $F$  is an odd deformation of  $A_{2/2}$ . By Theorems 3.9 and 3.16 we obtain that  $F$  is  $\mathcal{R}^{odd}$ -equivalent to  $\mathcal{R}^{odd}$ -versal deformation of  $A_{2/2} : (\beta, p, q) \mapsto \beta^3 + p\beta$ , since  $\frac{\partial F}{\partial p}(\beta, 0, 0) = -\beta$ .

If  $S \in \mathcal{M}_1^4$  and  $\frac{d^5 S}{dq^5}(0) \neq 0$  then  $\frac{\partial^k F}{\partial \beta^k}(0, 0, 0) = 0$  for  $k < 5$  and  $\frac{\partial^5 F}{\partial \beta^5}(0, 0, 0) \neq 0$  and consequently  $F$  is an odd deformation of  $A_{4/2}$ . By direct calculation,  $\frac{\partial^{k+1} F}{\partial \beta^k \partial q}(\beta, 0, 0) = \frac{1}{2} \left( \frac{d^{k+1} S}{dq^{k+1}}(\beta) + (-1)^{k+1} \frac{d^{k+1} S}{dq^{k+1}}(-\beta) \right)$ .

Then  $\frac{\partial^{k+1} F}{\partial \beta^k \partial q}(0, 0, 0) = 0$  for  $k < 3$  and  $\frac{\partial^4 F}{\partial \beta^3 \partial q}(0, 0, 0) = \frac{d^4 S}{dq^4}(0)$ . But  $\frac{\partial F}{\partial p}(\beta, 0, 0) = -\beta$ . So if  $\frac{d^4 S}{dq^4}(0) \neq 0$  we obtain by Theorem 3.9 and Corollary 3.16 that  $F$  is  $\mathcal{R}^{odd}$ -equivalent to  $\mathcal{R}^{odd}$ -miniversal deformation of  $A_{4/2} : (\beta, p, q) \mapsto \beta^5 + q\beta^3 + p\beta$ .  $\square$

**Corollary 4.8. (Geometric interpretation)** *If the curvature of the germ of a Lagrangian curve  $L$  does not vanish at  $(p_0, q_0) \in L$ , then the germ at  $(p_0, q_0)$  of the Wigner caustic on shell consists of  $L$  only and is Lagrangian stable. All germs of Wigner caustics of Lagrangian curves at such points are Lagrangian equivalent.*

*If, at  $(p_0, q_0) \in L$ , the curvature of the germ of a Lagrangian curve  $L$  vanishes but the first and the second derivatives of the curvature do*

not vanish, then the germ at  $(p_0, q_0)$  of the Wigner caustic on shell consists of two components:  $L$  and the germ at  $(p_0, q_0)$  of a 1-dimensional smooth submanifold with boundary  $(p_0, q_0)$ , which is 1-tangent to  $L$  at  $(p_0, q_0)$  and is simple stable. Any germ of the Wigner caustic in such a point is Lagrangian equivalent to the following germ at 0:

$$\{(p, q) \in \mathbb{R}^2 : p = 0\} \cup \left\{ (p, q) \in \mathbb{R}^2 : p = -\frac{27}{50}q^2, q \leq 0 \right\}.$$

The germs of the Wigner caustics of  $L$  at points of  $L$  which do not satisfy the above conditions are not stable.

*Proof.* This is an obvious corollary of Theorem 4.7, because the curvature of a curve  $L$  described by (2.7),  $i = 1$ , is given by  $\kappa\left(\frac{dS}{dq}(q), q\right) = \frac{d^3S}{dq^3}(q) / \left(1 + \left(\frac{d^2S}{dq^2}(q)\right)^2\right)^{3/2}$ . Thus  $\kappa(p_0, q_0) = \frac{d^3S}{dq^3}(q_0)$  since  $\frac{d^2S}{dq^2}(q_0) = 0$ . If  $\kappa(p_0, q_0) = 0$  then  $\frac{d\kappa}{dq}(p_0, q_0) = \frac{d^4S}{dq^4}(q_0)$  and  $\frac{d^2\kappa}{dq^2}(p_0, q_0) = \frac{d^5S}{dq^5}(q_0)$   $\square$

**Remark 4.9.** Although the curvature of a plane curve is not an affine invariant, the vanishing or not vanishing of the curvature is an affine invariant. Also, where the curvature is zero, the vanishing or not vanishing of its first two derivatives is also an affine invariant. Thus, Corollary 4.8 provides coordinate-free affine-symplectic invariant conditions for the realization of the singularities of the Wigner caustic on shell of a Lagrangian curve on the affine symplectic plane. Similar results for curves on a affine plane without a symplectic structure can be found in [8], where bifurcations of affine equidistants were studied.

**4.2. The Wigner caustic on shell of a Lagrangian surface.** Let  $L$  be the germ at 0 of a Lagrangian surface in symplectic affine space  $(\mathbb{R}^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$  and, without loss of generality, assume that  $L$  is generated by a function-germ  $S \in \mathcal{M}_2^3 \subset \mathcal{E}_2$  by (2.7),  $i = 2$ , and that  $F$  of form (2.13) is the generating family of  $\mathcal{L}$ .

**Notation 4.10.** To simplify the equations, we use the following:

$$S_{i,j} = \frac{\partial^{i+j} S}{\partial q_1^i \partial q_2^j}(0, 0), \quad S_{i,j}(q) = \frac{\partial^{i+j} S}{\partial q_1^i \partial q_2^j}(q_1, q_2).$$

Then, the 3-jet of  $S$  at 0 has the form

$$j_0^3 S = \frac{1}{6} S_{3,0} q_1^3 + \frac{1}{2} S_{2,1} q_1^2 q_2 + \frac{1}{2} S_{1,2} q_1 q_2^2 + \frac{1}{6} S_{0,3} q_2^3$$

and the **discriminant** of  $j_0^3 S$  has the following form  $\Delta(j_0^3 S) =$

$$\frac{1}{48} (3S_{1,2}^2 S_{2,1}^2 - 4S_{0,3} S_{2,1}^3 - 4S_{1,2}^3 S_{3,0} - S_{0,3}^2 S_{3,0}^2 + 6S_{0,3} S_{1,2} S_{2,1} S_{3,0})$$

**Theorem 4.11.** *If  $\Delta(j_0^3 S) > 0$ ,  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $D_{4/2}^- : (\beta_1, \beta_2, p, q) \mapsto \beta_1^2 \beta_2 - \beta_2^3 + p_1 \beta_1 + p_2 \beta_2$ .*

*If  $\Delta(j_0^3 S) < 0$ ,  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $D_{4/2}^+ : (\beta_1, \beta_2, p, q) \mapsto \beta_1^2 \beta_2 + \beta_2^3 + p_1 \beta_1 + p_2 \beta_2$ .*

*Proof.* By (2.15) we get that  $j_0^3 f = j_0^3 S$ . If  $\Delta(j_0^3 S) > 0$ , by linear change of coordinates we can reduce  $j_0^3 f$  to  $\beta_1^2 \beta_2 - \beta_2^3$ . Then repeating the arguments in the proof of Theorem 3.17 it is easy to see that  $f$  is  $\mathcal{R}^{odd}$ -equivalent to  $D_{4/2}^-$  singularity. By Theorem 4.4 it is easy to see that (2.13) is an  $\mathcal{R}^{odd}$ -versal deformation of  $f$ . By Corollary 3.18 we get the result. The case  $\Delta(j_0^3 S) < 0$  is analogous.  $\square$

**Lemma 4.12.**  $S_{3,0}S_{1,2} - S_{2,1}^2 \leq 0$  and  $S_{0,3}S_{2,1} - S_{1,2}^2 \leq 0$ , if  $\Delta(j_0^3 S) = 0$ .

*Proof.* The condition  $\Delta(j_0^3 S) = 0$  implies that  $w(t) = \frac{1}{6}S_{3,0}t^3 + \frac{1}{2}S_{2,1}t^2 + \frac{1}{2}S_{1,2}t + \frac{1}{6}S_{0,3}$  and  $v(t) = \frac{1}{6}S_{3,0} + \frac{1}{2}S_{2,1}t + \frac{1}{2}S_{1,2}t^2 + \frac{1}{6}S_{0,3}t^3$  have real roots of multiplicity greater than 1. Thus polynomials  $\frac{dw}{dt}(t) = \frac{1}{2}S_{3,0}t^2 + S_{2,1}t + \frac{1}{2}S_{1,2}$  and  $\frac{dv}{dt}(t) = \frac{1}{2}S_{2,1} + S_{1,2}t + \frac{1}{2}S_{0,3}t^2$  have real roots. So their discriminants are nonnegative.  $\square$

**Notation 4.13.** Now we introduce the following abbreviations:

$$\begin{aligned} r_1 &= \frac{S_{2,1}S_{1,2} - S_{3,0}S_{0,3}}{2(S_{3,0}S_{1,2} - S_{2,1}^2)}, \quad r_2 = \frac{S_{3,0}^2S_{0,3} - S_{3,0}S_{2,1}S_{1,2} + 3S_{2,1}^3}{S_{3,0}S_{1,2} - S_{2,1}^2} \\ \sigma_{0,n} &= \frac{\sum_{k=0}^n \binom{n}{k} S_{k,n-k} r_1^k}{(S_{3,0}r_1 - r_2)^n} \quad \text{for } n = 5, 7 \\ \tilde{r}_1 &= \frac{S_{2,1}S_{1,2} - S_{3,0}S_{0,3}}{2(S_{0,3}S_{2,1} - S_{1,2}^2)}, \quad \tilde{r}_2 = \frac{S_{0,3}^2S_{3,0} - S_{0,3}S_{1,2}S_{2,1} + 3S_{1,2}^3}{S_{0,3}S_{2,1} - S_{1,2}^2} \\ \sigma_{n,0} &= \frac{\sum_{k=0}^n \binom{n}{k} S_{n-k,k} \tilde{r}_1^k}{(S_{0,3}\tilde{r}_1 - \tilde{r}_2)^n} \quad \text{for } n = 5, 7 \end{aligned}$$

**Theorem 4.14.** *Assume  $S$  satisfies condition (4.1) and  $\Delta(j_0^3 S) = 0$ . Consider the following pair of conditions:*

$$(4.3) \quad S_{3,0}S_{1,2} - S_{2,1}^2 < 0,$$

$$(4.4) \quad S_{0,3}S_{2,1} - S_{1,2}^2 < 0.$$

*If (4.3) is satisfied and  $\sigma_{0,5} > 0$ , or (4.4) is satisfied and  $\sigma_{5,0} > 0$ , then  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $D_{6/2}^+ : (\beta_1, \beta_2, p, q) \mapsto \beta_1^2 \beta_2 + \beta_2^5 + p_1 \beta_1 + p_2 \beta_2 + q_1 \beta_2^3$ .*

*If (4.3) is satisfied and  $\sigma_{0,5} < 0$ , or (4.4) is satisfied and  $\sigma_{5,0} < 0$ , then  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $D_{6/2}^- : (\beta_1, \beta_2, p, q) \mapsto \beta_1^2 \beta_2 - \beta_2^5 + p_1 \beta_1 + p_2 \beta_2 + q_1 \beta_2^3$ .*

*Proof.* First we assume  $\Delta(j_0^3 S) = 0$  and condition (4.3), with  $\sigma_{0,5} > 0$ . Then we get  $j_0^3 f = j_0^3 S = (\beta_1 - r_1 \beta_2)^2 (S_{3,0} \beta_1 - r_2 \beta_2) = \tilde{\beta}_1^2 \tilde{\beta}_2$ , where  $(\tilde{\beta}_1, \tilde{\beta}_2) = (\beta_1 - r_1 \beta_2, S_{3,0} \beta_1 - r_2 \beta_2)$  forms the coordinate system on  $\mathbb{R}^2$ , since by condition (4.3)  $r_1 \neq r_2 / S_{3,0}$ .  $\sigma_{0,5} > 0$  is equivalent to  $\frac{\partial^5 f}{\partial \beta_2^5}(0) > 0$ . Thus,  $f$  is  $\mathcal{R}^{odd}$ -equivalent to  $D_{6/2}^+$ . By Theorem 4.4 we obtain that  $F$  is an  $\mathcal{R}^{odd}$ -versal deformation of  $f$  since  $S$  satisfies (4.1). If  $\Delta(j_0^3 S) = 0$  and (4.4) is satisfied with  $\sigma_{5,0} > 0$ , then we repeat in the same way using the coordinate system  $(\tilde{\beta}_1, \tilde{\beta}_2) = (\beta_2 - \tilde{r}_1 \beta_1, S_{0,3} \beta_2 - \tilde{r}_2 \beta_1)$ . The cases (4.3) and  $\sigma_{0,5} < 0$ , or (4.4) and  $\sigma_{5,0} < 0$ , are analogous.  $\square$

**Theorem 4.15.** *Assume  $S$  satisfies condition (4.1) and  $\Delta(j_0^3 S) = 0$ .*

*If (4.3) holds,  $\sigma_{0,5} = 0$  and  $\sigma_{0,7} > 0$ , or, if (4.4) holds,  $\sigma_{5,0} = 0$  and  $\sigma_{7,0} > 0$ , then  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $D_{8/2}^+ : (\beta_1, \beta_2, p, q) \mapsto \beta_1^2 \beta_2 + \beta_2^7 + p_1 \beta_1 + p_2 \beta_2 + q_1 \beta_2^3 + q_2 \beta_2^5$ .*

*If (4.3) holds,  $\sigma_{0,5} = 0$  and  $\sigma_{0,7} < 0$ , or, if (4.4) holds,  $\sigma_{5,0} = 0$  and  $\sigma_{7,0} < 0$ , then  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $D_{8/2}^- : (\beta_1, \beta_2, p, q) \mapsto \beta_1^2 \beta_2 - \beta_2^7 + p_1 \beta_1 + p_2 \beta_2 + q_1 \beta_2^3 + q_2 \beta_2^5$ .*

*Proof.* First assume  $\Delta(j_0^3 S) = 0$ , condition (4.3) is satisfied and  $\sigma_{0,5} = 0$ ,  $\sigma_{0,7} > 0$ . As in the proof of Theorem 4.14 we get  $j_0^3 f = j_0^3 S = \tilde{\beta}_1^2 \tilde{\beta}_2$ , where  $(\tilde{\beta}_1, \tilde{\beta}_2) = (\beta_1 - r_1 \beta_2, S_{3,0} \beta_1 - r_2 \beta_2)$  and  $\frac{\partial^5 f}{\partial \beta_2^5}(0) = 0$  and  $\frac{\partial^7 f}{\partial \beta_2^7}(0) > 0$ , since  $\sigma_{0,5} = 0$  and  $\sigma_{0,7} > 0$ . Thus  $f$  is  $\mathcal{R}^{odd}$ -equivalent to  $D_{8/2}^+$ . By Theorem 4.4,  $F$  is an  $\mathcal{R}^{odd}$ -versal deformation of  $f$  since  $S$  satisfies (4.1). If  $\Delta(j_0^3 S)$  vanishes, condition (4.4) is satisfied and  $\sigma_{5,0} = 0$ ,  $\sigma_{7,0} > 0$ , we repeat using the coordinate system  $(\tilde{\beta}_1, \tilde{\beta}_2) = (\beta_2 - \tilde{r}_1 \beta_1, S_{0,3} \beta_2 - \tilde{r}_2 \beta_1)$ . The case (4.3),  $\sigma_{0,5} = 0$  and  $\sigma_{0,7} < 0$ , and the case (4.4),  $\sigma_{5,0} = 0$  and  $\sigma_{7,0} < 0$ , are worked out analogously.  $\square$

**Theorem 4.16.** *Assume  $S$  satisfies condition (4.1) and  $\Delta(j_0^3 S) = 0$ .*

*If either of the following two conditions are satisfied,*

(4.5)

$$S_{3,0} S_{1,2} - S_{2,1}^2 = 0, \quad S_{3,0} \neq 0, \quad \sum_{k=0}^5 \binom{5}{k} S_{k,5-k} (-S_{2,1})^k (S_{3,0})^{5-k} \neq 0,$$

(4.6)

$$S_{0,3} S_{2,1} - S_{1,2}^2 = 0, \quad S_{0,3} \neq 0, \quad \sum_{k=0}^5 \binom{5}{k} S_{5-k,k} (-S_{1,2})^k (S_{0,3})^{5-k} \neq 0,$$

*then  $F$  is  $\mathcal{R}^{odd}$ -equivalent to the  $\mathcal{R}^{odd}$ -versal deformation of  $E_{8/2} : (\beta_1, \beta_2, p, q) \mapsto \beta_1^3 + \beta_2^5 + p_1 \beta_1 + p_2 \beta_2 + q_1 \beta_1 \beta_2^2 + q_2 \beta_2^3$ .*

*Proof.* First we assume that  $\Delta(j_0^3 S) = 0$  and condition (4.5) is satisfied. It implies that we get  $j_0^3 f = j_0^3 S = \tilde{\beta}_1^3$ , where  $(\tilde{\beta}_1, \tilde{\beta}_2) = \left( \left( \frac{S_{3,0}}{6} \right)^{1/3} \left( \beta_1 - \frac{S_{2,1}}{S_{3,0}} \beta_2 \right), \beta_2 \right)$  and  $\frac{\partial^5 f}{\partial \beta_2^5}(0) \neq 0$ . Thus  $f$  is  $\mathcal{R}^{odd}$ -equivalent to  $E_{8/2}$ . By Theorem 4.4 we obtain that  $F$  is an  $\mathcal{R}^{odd}$ -versal deformation of  $f$  since  $S$  satisfies (4.1). If  $\Delta(j_0^3 S) = 0$  and condition (4.6) is satisfied, we repeat with  $(\tilde{\beta}_1, \tilde{\beta}_2) = \left( \left( \frac{S_{0,3}}{6} \right)^{1/3} \left( \beta_2 - \frac{S_{1,2}}{S_{0,3}} \beta_1 \right), \beta_1 \right)$ .  $\square$

**Remark 4.17.** These are all odd-simple singularities that can be realized as singularities of on-shell Wigner caustics of Lagrangian submanifolds in affine-symplectic space. The odd-simple singularities  $J_{10/2}^\pm$  and  $E_{12/2}$  cannot be realized in this way because their codimensions are too big for a Lagrangian surface in affine-symplectic 4-space. On the other hand, for higher dimensional Lagrangian submanifolds in affine-symplectic space, the necessary number of variables for the generating families of on-shell Wigner caustics is at least 3 (see Remark 3.14).

**4.3. Geometric interpretation.** Finally, we provide the geometric interpretation of each realization condition for simple stable Lagrangian singularities of on-shell Wigner caustics of Lagrangian surfaces. This also provides affine-invariant descriptions for such realization conditions, which were presented in a particular coordinate system, in Theorems 4.11-4.16. Similar results for surfaces on a affine 4-space without a symplectic structure can be found in [10], where geometry of surfaces through the contact map was studied. Background for extrinsic geometry of surfaces in euclidean 4-space can be found in [12]. Here, we merely adapt it to the case of Lagrangian surfaces in affine-symplectic 4-space. Recall Notation 4.10.

Then, for the canonical euclidean metric in  $\mathbb{R}^4$ , the matrix of the second fundamental form at  $(p, q)$  of  $L$  can be written as follows:

$$II_{(p,q)} = \begin{bmatrix} S_{3,0}(q) & S_{2,1}(q) & S_{1,2}(q) \\ S_{2,1}(q) & S_{1,2}(q) & S_{0,3}(q) \end{bmatrix}$$

from which is defined the following determinant:

$$\Delta_L(p, q) = \frac{1}{4} \det \begin{bmatrix} S_{3,0}(q) & 2S_{2,1}(q) & S_{1,2}(q) & 0 \\ 0 & S_{3,0}(q) & 2S_{2,1}(q) & S_{1,2}(q) \\ S_{2,1}(q) & 2S_{1,2}(q) & S_{0,3}(q) & 0 \\ 0 & S_{2,1}(q) & 2S_{1,2}(q) & S_{0,3}(q) \end{bmatrix}$$

and it is easy to see that

$$(4.7) \quad \Delta_L(p, q) = -16\Delta(j_q^3 S).$$

Also, the Gaussian curvature at  $(p, q) \in L$  is given by the formula

$$(4.8) \quad \kappa(p, q) = S_{3,0}(q)S_{1,2}(q) - (S_{2,1}(q))^2 + S_{2,1}(q)S_{0,3}(q) - (S_{1,2}(q))^2.$$

In extrinsic geometry of surfaces in euclidean  $\mathbb{R}^4$ ,  $\Delta_L$  and  $\kappa$  are both invariant under the action of the euclidean group of isometries on  $\mathbb{R}^4$ , but neither is invariant under the action of the whole affine group on  $\mathbb{R}^4$ . The same is true if we restrict to the action of the affine-symplectic group on symplectic  $\mathbb{R}^4$ . However, although neither  $\Delta_L$  nor  $\kappa$  are affine-symplectic invariants, the following propositions allow us to use them for classifying points in a Lagrangian surface of symplectic  $\mathbb{R}^4$ .

**Proposition 4.18.** *The sign ( $> 0$ ,  $< 0$ ,  $= 0$ ) of  $\Delta_L$  is an affine (and therefore affine-symplectic) invariant.*

*Proof.* The proof follows from the following two statements:

- (i) The sign of  $\Delta_L$  stratifies the singularities of height functions  $h[\iota] : L \times S^3 \rightarrow \mathbb{R}$ ,  $(m, v) \mapsto \langle \iota(m), v \rangle$ , where  $\iota : L \rightarrow \mathbb{R}^4$  is an embedding,  $S^3 \subset \mathbb{R}^4$  is the unit sphere, and  $\langle \cdot, \cdot \rangle$  is the euclidean inner product in  $\mathbb{R}^4$  (see [12], Lemma 3.2, which relates the sign of  $\Delta_L(p, q)$  to the number of unit vectors  $v$  normal to  $L$  at  $(p, q) \in L$  for which  $(p, q)$  is a degenerate critical point of the height function  $h[\iota, v] : L \rightarrow \mathbb{R}$ ).
- (ii) The stratification of the singularities of height functions  $h[\iota]$  is invariant under affine transformations (see [4], Proposition A.4, which relates singularities of height functions to contact with hyperplanes and shows that the stratification of these contacts is affine invariant).  $\square$

Recall that a point  $(p, q) \in L$  is called: (i) **parabolic** if  $\Delta_L(p, q) = 0$ , (ii) **elliptic** if  $\Delta_L(p, q) > 0$ , (iii) **hyperbolic** if  $\Delta_L(p, q) < 0$ , see [12].

From Proposition 4.18, such a classification of points on  $L \subset \mathbb{R}^4$  (with symplectic structure) is affine (and therefore affine-symplectic) invariant and, from equation (4.7), we obtain the following immediate corollary of Theorem 4.11, which gives a geometrical characterization of singularities  $D_{4/2}^\pm$  of the Wigner caustic on shell.

**Corollary 4.19.** *Let  $L$  be a Lagrangian surface. Iff  $(p, q) \in L$  is a hyperbolic point, the germ of Wigner caustic on shell at  $(p, q)$  is generated by function-germ of type  $D_{4/2}^-$  and it consists of  $L$  only, being simple stable. Iff  $(p, q) \in L$  is an elliptic point, the germ of Wigner caustic on shell is generated by function-germ of type  $D_{4/2}^+$  and is Lagrangian equivalent to the following simple stable germ at 0:*

$$\mathbf{E}_{1/2}(L) = \{(p, q) \in \mathbb{R}^4 : 3p_1^2 = p_2^2, p_2 \leq 0\}.$$

We also recall [12] that a *parabolic* point  $m \in M^2 \subset \mathbb{R}^4$  is called

- (i-i) an *inflection point of imaginary type*, if  $\kappa(m) > 0$ ,
- (i-ii) an *inflection point of real type*, if  $\kappa(m) < 0$ ,  $\text{rank}\{II_{(m)}\} = 1$ ,
- (i-iii) a *point of nondegenerate ellipse*, if  $\kappa(m) < 0$ ,  $\text{rank}\{II_{(m)}\} = 2$ ,
- (i-iv) an *inflection point of flat type*, if  $\kappa(m) = 0$ .

Again, we refer to [12] where the above classification of parabolic points on  $M^2 \subset \mathbb{R}^4$  is related to the classification of singularities of height functions, which, from Proposition A.4 in [4] implies:

**Proposition 4.20.** *When  $\Delta(p, q) = 0$ , the classification of the parabolic point  $m = (p, q) \in M^2 \subset \mathbb{R}^4$  (with symplectic structure) given by (i-i)-(i-iv) above is affine (and therefore affine-symplectic) invariant.*

And thus, finally, we obtain the other geometric characterizations.

**Corollary 4.21.** *If  $L$  is a Lagrangian surface, then  $L$  has no inflection points of real or imaginary types. The germ of the Wigner caustic on shell at  $(p, q) \in L$  has simple stable  $\mathbb{Z}_2$ -symmetric singularity generated by function-germ of type  $D_{6/2}^\pm$  or  $D_{8/2}^\pm$  only if  $(p, q)$  is a parabolic point of nondegenerate ellipse, and by function-germ of type  $E_{8/2}$  only if  $(p, q)$  is an inflection point of flat type.*

*Proof.* (4.8) and Lemma 4.12 imply that if  $(p, q)$  is a parabolic point of a Lagrangian surface  $L$  then the Gaussian curvature  $\kappa(p, q)$  is nonpositive. Thus,  $L$  has no inflection points of imaginary type. The simple observation, that, in the Lagrangian case (only), if  $\text{rank}\{II_{(p,q)}\} = 1$  then  $\kappa(p, q) = 0$ , implies  $L$  has no inflection points of real type. The second statement follows from Theorems 4.14-4.15-4.16 and the fact that  $\Delta(j_0^3 S) = 0$  together with one of the conditions  $S_{3,0}S_{1,2} - S_{2,1}^2 = 0$ ,  $S_{3,0} \neq 0$ , or  $S_{0,3}S_{2,1} - S_{1,2}^2 = 0$ ,  $S_{0,3} \neq 0$ , imply  $\kappa(p, q) = 0$ .  $\square$

**Remark 4.22.** Simply saying that  $(p, q) \in L$  is a parabolic point of nondegenerate ellipse is not enough to characterize the type of singularity of the Wigner caustic on shell at  $(p, q)$ . Therefore, for a Lagrangian surface, the type of singularity of the Wigner caustic on shell at a parabolic point of nondegenerate ellipse is a further affine-symplectic invariant that allows for a finer classification of the point.

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